BASE CHANGE MAPS FOR UNIPOTENT ALGEBRA GROUPS

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Contents

Introduction

Let p be a fixed prime, and let $\mathbb{F}$ be a fixed algebraic closure of a finite field with p elements. If q is a power of p, we will denote by $\mathbb{F}_q$ the unique subfield of $\mathbb{F}$ that consists of q elements. We will denote by Fr_q, or simply Fr if no confusion is possible, the canonical topological generator of the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, given by $x \mapsto x^q$. By a *representation* of a finite group we will always mean a complex finite dimensional representation.

Let G be a unipotent algebraic group over $\mathbb{F}_q$. For every $n \in \mathbb{N}$, let $\Gamma_n = G(\mathbb{F}_{q^n})$, a finite nilpotent group. We have the obvious action of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ on Γ_n, and hence an induced action of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ on the set $\widehat{\Gamma}_n$ of isomorphism classes of its irreducible representations. The three basic questions in geometric character theory for unipotent groups over finite fields are as follows (cf. [Dri05], which was motivated in part by [Lus03]):

(1) Do there exist "base change maps"

$$T_m^n : \widehat{\Gamma}_m \longrightarrow \widehat{\Gamma}_n^{\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_{q^m})}$$

for all $m, n \in \mathbb{N}$ such that $m|n$, which are Fr-equivariant and satisfy

$$T_m^k = T_n^k \circ T_m^n$$

for all $m, n, k \in \mathbb{N}$ such that $m|n|k$?

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(2) Assuming that the answer to question (1) is positive, are the base change maps injective (resp., surjective; resp., bijective)?

(3) Assuming again that the answer to question (1) is positive, form the direct limit

$$\widehat{\Gamma} := \varinjlim \widehat{\Gamma}_n$$

with respect to the base change maps, and equip it with the induced action of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. Does there exist a "geometric object" $\widehat{G}$ defined over $\mathbb{F}_q$ such that its set of geometric points $\widehat{G}(\mathbb{F})$ admits an Fr-equivariant bijection onto $\widehat{\Gamma}$?

We prefer not to make the last question precise at this point.

In this paper we study the first two questions for a special class of groups. We recall that a *finite algebra group* (cf. [Isa95]) is a group of the form $1 + A$, where A is a finite dimensional associative nilpotent algebra over $\mathbb{F}_q$, and $1+A$ consists of formal expressions of the form $1+a$, where $a \in A$, multiplied in the natural way: $(1+a)(1+b) = 1+(a+b+ab)$. It is clear that one can also think of $G = 1 + A$ as an affine algebraic group over $\mathbb{F}_q$: the underlying variety is the affine space over $\mathbb{F}_q$ underlying the algebra A, and for any commutative k-algebra R, we have $G(R) = 1 + R \otimes_k A$. When thought of in this way, we will call G a *unipotent algebra group* to emphasize that we consider it as a geometric, rather than algebraic, object. We will prove the following

Theorem. *There exist canonically defined injective base change maps for irreducible representations of finite algebra groups.*

The main technical tool used in the proof is the notion of a "strongly Heisenberg representation" of a finite algebra group, introduced below. Apart from its significance in the geometric theory, this notion also seems to have applications to purely algebraic questions: for example, it can be used to rewrite Halasi's proof [Hal04] of Gutkin's conjecture [Gut73] (see Section 2) in such a way that it no longer uses Isaacs' Theorem A [Isa95].

It is interesting to note that the only previously known examples of base change maps for unipotent groups over finite fields arise from situations where there exists an explicit classification of irreducible representations of the groups Γ_n in the spirit of the orbit method and/or Mackey's little groups method. For instance, an analogue of Kirillov's orbit method bijection can be used to define natural base change maps when G is a connected unipotent group over $\mathbb{F}_q$ that has nilpotence class less than p. In all such situations, one effectively starts from a positive answer to question (3), and uses it to answer questions (1) and (2). On the other hand, question (3) for unipotent algebra groups is still open, and no explicit classification of irreducible representations is known. It is also worth noting that in the situations where the orbit method does work, the base change maps one obtains are necessarily surjective, but may fail to be injective. On the other hand, the base change maps constructed in our paper are always injective. We do not know if they are always surjective, but we expect that the answer is negative, even though it is positive in many special cases.

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1. Strongly Heisenberg representations

1.1. **Heisenberg representations.** Once and for all, we make the convention that round parentheses $(\cdot,\cdot)$ denote the group commutator, while square brackets $[\cdot,\cdot]$ denote the Lie bracket (which in particular is well defined in any associative algebra). In order to motivate the definition of a strongly Heisenberg representation, we first recall the more familiar notion of a Heisenberg representation. We say that an irreducible representation $\rho : G \to GL(V)$ of a finite group G is a *Heisenberg representation* if there exists a subgroup $H \subseteq G$ such that G/H is abelian and H acts on V by scalars, i.e., $\rho(H) \subseteq \mathbb{C}^\times$. Of course, if ρ is Heisenberg, then it is clear that one can take $H = (G, G)$ in the definition, but it is often convenient to allow H to be larger. It is easy to check that the following conditions are equivalent:

(i) ρ is Heisenberg;
(ii) the image $\rho(G) \subseteq GL(V)$ is a nilpotent group of nilpotence class at most 2;
(iii) $G/Z(\rho)$ is abelian, where the center $Z(\rho)$ of ρ is defined as

$$Z(\rho) = \{g \in G \big| (\rho(g), \rho(h)) = 1 \ \forall\, h \in G\} = \{g \in G \big| \rho(g) \in \mathbb{C}^\times\}.$$

We now recall a basic fact. Let G be an arbitrary finite group, let $\rho : G \longrightarrow GL(V)$ be a Heisenberg representation, and let $H \subseteq G$ be a subgroup such that G/H is abelian and $\rho(H) \subseteq \mathbb{C}^\times$. The following result is well known (and easy to prove).

Lemma 1.1. *The commutator pairing*

$$c_\rho : G/H \times G/H \longrightarrow \mathbb{C}^\times$$

given by

$$c_\rho(\overline{g}_1, \overline{g}_2) \cdot \mathrm{id}_V = \rho(g_1 g_2 g_1^{-1} g_2^{-1})$$

is well defined and yields an alternating bilinear form on the finite abelian group G/H. If $L \subseteq G/H$ is any maximal isotropic subgroup with respect to c_ρ, and $\widetilde{L} \subseteq G$ is its preimage in G, then ρ is induced from a one-dimensional representation of $\widetilde{L}$. Furthermore, ρ is determined, up to isomorphism, by the character via which it acts on $Z(\rho)$.

1.2. **Strongly Heisenberg representations.** Let $k \subset \mathbb{F}$ be a finite subfield, and let $G = 1 + A$ be a finite algebra group over k.

Definition 1.2. An irreducible representation

$$\rho : 1 + A \longrightarrow GL(V)$$

is said to be *strongly Heisenberg* if the subgroup $1 + A^2 \subset 1 + A$ acts by scalars on V, i.e.,

$$\rho(1 + A^2) \subseteq \mathbb{C}^\times \subseteq GL(V).$$

Let ρ be a strongly Heisenberg representation of a finite algebra group $G = 1 + A$, and take $H{=}1 + A^2$. It is clear that G/H is isomorphic to the underlying additive group of A/A^2, and in particular is abelian, whence every strongly Heisenberg representation is, a fortiori, Heisenberg. On the other hand, A/A^2 is also a vector space over k, and we have the following main result of the section:

Proposition 1.3. *In this situation, with the notation of Lemma 1.1, there exists a maximal isotropic subgroup $L \subseteq G/H = A/A^2$ with respect to c_ρ which is also a k-subspace of A/A^2. Consequently, $\widetilde{L}$ is an algebra subgroup of G.*

The proof can be easily reduced to the following

Lemma 1.4. *Let $\phi : 1 + A^2 \to \mathbb{C}^\times$ be a G-invariant character. The commutator pairing*

$$c = c_\phi : (1 + A)/(1 + A^2) \times (1 + A)/(1 + A^2) \longrightarrow \mathbb{C}^\times,$$

given by $c(\overline{g}, \overline{h}) = \phi((g, h))$, is well defined and satisfies the equation

$$c(1 + \lambda a, 1 + b) = c(1 + a, 1 + \lambda b) \qquad \forall\, a, b \in A,\ \lambda \in k. \tag{1.1}$$

Indeed, assume that this lemma has been proved. In the notation of Proposition 1.3, if ϕ is the character by which ρ acts on $1 + A^2$, it is clear that $c_\rho = c_\phi$. Now let $L \subseteq A/A^2$ be a k-subspace which is maximal among all k-subspaces that are isotropic with respect to c_ρ. Equation (1.1) implies that the annihilator $L^\perp$ of L with respect to c_ρ is also a k-subspace of A/A^2. If $L \neq L^\perp$, then (1.1) also shows that L together with any element of $L^\perp \setminus L$ spans a k-subspace of A/A^2 which is isotropic with respect to c_ρ, contradicting our choice of L. This completes the proof of Proposition 1.3.

1.3. **Commutators in algebra groups.** In our proof of Lemma 1.4 we will need the following result.

Proposition 1.5 (Halasi)**.** *Let R be an integral domain whose quotient field has characteristic zero, and let $J = F_R(n, X)$ be a free associative nilpotent algebra of class $n \in \mathbb{N}$ generated by a set X over R. For all $k \geq 2$, we have*

$$(1 + J, 1 + J) \cap (1 + J^k) = (1 + J, 1 + J^{k-1}).$$

Formally, $F_R(n, X)$ denotes the initial object in the category of all maps from the set X to associative nilpotent algebras A over R satisfying $A^n = (0)$. The proposition is proved in [Hal04] in the special case $R = \mathbb{Z}$; however, it is easy to check that all steps in Halasi's proof work equally well for an arbitrary integral domain of characteristic zero.

1.4. **Proof of Lemma 1.4.** The G-invariance of ϕ is equivalent to $\phi\big((1+A,1+A^2)\big)=\{1\}$. The fact that c is well defined is obvious from this. For the second statement of the lemma, it suffices to prove that ϕ annihilates all products of the form $(1+\lambda a,1+b)\cdot(1+a,1+\lambda b)^{-1}$. Now a straightforward computation shows that $(1+\lambda a,1+b)\cdot(1+a,1+\lambda b)^{-1}\in 1+A^3$, and, of course, we also have $(1+\lambda a,1+b)\cdot(1+a,1+\lambda b)^{-1}\in(1+A,1+A)$. We would be done if we could apply Proposition 1.5 at this point. In order to be able to use it, let $n\in\mathbb{N}$ be such that $A^n=(0)$, let R be an arbitrary integral domain of characteristic zero which admits a surjective homomorphism $R\to k$, let $\widetilde{\lambda}\in R$ be a lift of λ, and let $J=F_R(n,\{X,Y\})$ denote the free associative nilpotent algebra of nilpotence class n generated by two elements, X and Y, over R. Then, by Proposition 1.5,

$$(1+\widetilde{\lambda}X,1+Y)\cdot(1+X,1+\widetilde{\lambda}Y)^{-1}\in(1+J,1+J)\cap(1+J^3)=(1+J,1+J^2).$$

There exists a homomorphism of R-algebras $J\to A$ (where A is viewed as an R-algebra via $R\to k$) which takes $X\mapsto a$ and $Y\mapsto b$. Since it clearly takes $(1+J,1+J^2)$ into $(1+A,1+A^2)$, the proof is complete.

2. Theorems of Gutkin, Halasi, and Isaacs

2.1. **Main result.** One of the main algebraic results about representations of finite algebra groups is the following

Theorem 2.1 (Gutkin-Halasi). *Every irreducible representation of a finite algebra group $1+A$ is induced from a one-dimensional representation of a subgroup of the form $1+B$, where $B\subseteq A$ is a subalgebra.*

It was stated by E.A. Gutkin in [Gut73]; however, to the best of our knowledge, the first complete proof was given by Z. Halasi in [Hal04]. The following statement is an obvious consequence of the theorem:

Corollary 2.2 (Isaacs). *If $G=1+A$ is a finite algebra group over $\mathbb{F}_q$, the dimension of every irreducible representation of G is a power of q.*

Historically, however, this corollary was proved by I.M. Isaacs before Theorem 2.1, using nontrivial techniques from character theory of finite groups; it appears as Theorem A in [Isa95]. Moreover, Halasi's proof uses Isaacs' result in an essential way; on the other hand, Halasi's proof is more "Lie-theoretic" than "group-theoretic." In this section we suggest a self-contained proof of Theorem 2.1. It includes Halasi's argument, which we reproduce here for the reader's convenience, and also because the technique of the proof will be useful later on. However, our proof avoids Isaacs' Theorem A by making use of the notion of a strongly Heisenberg representation (which does not appear in [Isa95] and [Hal04]). The advantage of our approach (apart from the fact that it yields a shorter proof) is that now the only nontrivial results from character theory that we use are the well-known Frobenius reciprocity theorem and Mackey's irreducibility criterion.

2.2. **Proof of Theorem 2.1.** We will use the following result:

Theorem 2.3 ([Hal04], Theorem 1.3)**.** *Let A be a finite dimensional associative nilpotent algebra over $\mathbb{F}_q$, and let $G = 1 + A$. If ϕ is an irreducible character of $1 + A^2$ which is G-invariant, then ϕ is linear (i.e., has degree 1).*

Now, in the situation of Theorem 2.3, let $\rho : G \to GL(V)$ be an irreducible representation. We may assume that $\dim V > 1$, in which case $\rho\big|_{1+A^2}$ is not irreducible by Theorem 2.3. Our proof uses induction on $\dim_{\mathbb{F}_q} A$ (the case when $\dim_{\mathbb{F}_q} A = 1$ being obvious). Thus we may further assume that the statement of Theorem 2.1, and hence, a fortiori, the statement of Isaacs' Theorem A [Isa95], holds for all finite algebra groups $1 + B$ with $\dim B < \dim A$. Using the transitivity of induction of group representations, it suffices to show that $\rho \cong \operatorname{Ind}_H^G \phi$ for some proper algebra subgroup $H \subsetneq G$ and some representation ϕ of H.

Let $\psi : 1 + A^2 \to GL(W)$ be an irreducible constituent of $\rho\big|_{1+A^2}$, and let $H = 1 + U \supseteq 1 + A^2$ be a maximal algebra subgroup of G such that ψ can be extended to H. We have, by Frobenius reciprocity,

$$\left[\operatorname{Ind}_{1+A^2}^H \psi : \operatorname{Res}_H^G \rho\right] = \left[\psi : \rho\big|_{1+A^2}\right] \geq 1,$$

whence $\operatorname{Ind}_{1+A^2}^H \psi$ and $\operatorname{Res}_H^G \rho$ share at least one irreducible summand, call it ϕ. Since ψ can be extended to H, and since $1 + A^2$ is normal in H, it follows that ϕ is an extension of ψ. We also have, by construction,

$$\left[\rho : \operatorname{Ind}_H^G \phi\right] = \left[\operatorname{Res}_H^G \rho : \phi\right] \geq 1.$$

Hence, if we prove that $\operatorname{Ind}_H^G \phi$ is irreducible, the argument will be complete.

The proof uses Mackey's criterion. Since H is normal in G (because $H \supseteq 1 + A^2$ and $(1 + A)/(1 + A^2)$ is abelian), we must show that for any $x \in A$ such that $1 + x \notin H$, the representations ϕ and ϕ^x are nonisomorphic, where

$$\phi^x : H \longrightarrow GL(W)$$

is defined by

$$\phi^x(h) = \phi\big((1 + x)h(1 + x)^{-1}\big).$$

Fix $x \in A \setminus U$, and set $N_x = 1 + \mathbb{F}_q x + U$; this is of course an algebra subgroup of G such that $[N_x : H] = q$. Pick an arbitrary irreducible representation λ of N_x such that ϕ is a summand of $\operatorname{Res}_H^{N_x} \lambda$.

Up to this point we have closely followed the proof given in [Hal04]. Here Halasi applies Isaacs' Theorem A, which shows that both $\dim \phi$ and $\dim \lambda$ are powers of q. On the other hand,

$$\dim \phi < \dim \lambda \leq \dim\big(\operatorname{Ind}_H^{N_x} \phi\big) = q \cdot \dim \phi,$$

where the first inequality is due to the fact that ϕ cannot be extended to N_x by the choice of H. Thus the second inequality must be an equality, whence $\operatorname{Ind}_H^{N_x} \phi \cong \lambda$ is irreducible. Applying the converse of Mackey's criterion implies that $\phi^x \ncong \phi$ and completes the proof.

We now explain how the use of Isaacs' Theorem A can be avoided. Note that if $N_x \neq G$, then by induction we already know that Isaacs' Theorem A holds for H and for N_x, whence there is no problem. So let us assume that $N_x = G$, or, equivalently, $\dim(A/U) = 1$. We now use an idea that appeared in [Isa95].

Pick an arbitrary subspace $U' \subset A$ such that $U' \neq U$, $A^2 \subseteq U'$, and $\dim(A/U') = 1$. Then $H' = 1 + U'$ is also an algebra subgroup of G; moreover, $HH' = G$. Let $V = U \cap U'$ and $M = 1 + V = H \cap H'$; we have $[H' : M] = q$. If the representation $\phi_0 := \phi\big|_M$ extends to H', then ϕ_0, and hence, a fortiori, ψ is invariant under both H and H', and hence under G. Now Theorem 2.3 implies that ψ has dimension 1. Moreover, we see that $\rho\big|_{1+A^2}$ decomposes into a direct sum of copies of ψ, so $\rho(1 + A^2)$ consists of scalars, i.e., ρ is a strongly Heisenberg representation. But in this case, by Proposition 1.3 and Lemma 1.1, we already know that Theorem 2.1 is valid.

The only remaining case is where ϕ_0 does not extend to H'. In this case, we can apply Isaacs' Theorem A to the groups M and H', so Halasi's dimension counting argument given above shows that $\mathrm{Ind}_M^{H'} \phi_0$ is irreducible. Mackey's criterion implies that the stabilizer of ϕ_0 in H' is equal to M. A fortiori, if G^ϕ denotes the stabilizer of the isomorphism class of ϕ under the action of G, then $H \subseteq G^\phi$ and $G^\phi \cap H' = M$. Since $G = HH'$, these two conditions force $G^\phi = H$, and so $\mathrm{Ind}_H^G \phi$ is irreducible (again by Mackey's criterion), which finally completes the proof of Theorem 2.1.

3. Reduction process for finite algebra groups

3.1. In this section we show that any irreducible representation $\rho : G \to GL(V)$ of an algebra group $G = 1 + A$ over a finite field $k \subset \mathbb{F}$ can be represented as the induced representation of a strongly Heisenberg representation of an algebra subgroup in a canonical way. The precise meaning of this statement is as follows.

Theorem 3.1 (Reduction process)**.** *There exists a canonical (i.e., independent of any choices) decomposition of V into a direct sum of subspaces*

$$V = \bigoplus_{i \in I} V_i \tag{3.1}$$

satisfying the following properties.

(1) For each $g \in G$ and each $i \in I$, there exists $g(i) \in I$ with $g(V_i) \subseteq V_{g(i)}$.
(2) For some, and hence every, $i \in I$, the group $G_i = \{g \in G | g(i) = i\}$ is an algebra subgroup of G.
(3) For some, and hence every, $i \in I$, the representation of G_i in V_i is strongly Heisenberg.
(4) If $\rho' : G' \to GL(V')$ is an irreducible representation of another algebra group $G' = 1 + A'$ over k, if $\phi : A \to A'$ is an algebra isomorphism, and if $\psi : V \to V'$ is a vector space isomorphism which intertwines $\rho \circ \phi^{-1}$ and ρ', then ψ takes the canonical decomposition of V onto the canonical decomposition of V'.

We remark that, in the situation of the theorem, if we choose $i \in I$, then properties (1) and (2) imply that ρ is naturally isomorphic to $\operatorname{Ind}_{G_i}^{G} \rho_i$, where ρ_i denotes the representation of G_i in V_i. However, we prefer not to choose i, which yields a more invariant statement.

3.2. The proof of the theorem is easy using the techniques developed in Section 2. Namely, by induction on $\dim V$, it suffices to show that if ρ is not strongly Heisenberg, then there exists a *nontrivial* decomposition 3.1 which is independent of any choices and satisfies properties (1), (2) and (4), but not necessarily (3), of the theorem.

Suppose ρ is not strongly Heisenberg, and let I denote the set of isomorphism classes of the irreducible constituents of $\rho\big|_{1+A^2}$. From Theorem 2.3, it follows that $|I| > 1$. For each $i \in I$, let $V_i \subset V$ denote the sum of all irreducible subrepresentations of $\rho\big|_{1+A^2}$ corresponding to i. Then it is clear that we obtain a decomposition of V of the form (3.1) satisfying properties (1) and (4), and we only need to verify property (2). It follows from the more general

Lemma 3.2. *Let $G = 1 + A$ be an algebra group over $\mathbb{F}_q$, let $U \subseteq A$ be any subspace such that $A^2 \subseteq U$, so that $H = 1 + U$ is automatically a normal algebra subgroup of G, and let ψ be any irreducible representation of H. Then the stabilizer G^ψ of (the isomorphism class of) ψ in G is an algebra subgroup of G.*

Proof. Since G/H is isomorphic to the underlying additive group of A/U, and since $H \subseteq G^\psi$, it suffices to show that if $1+x \in G^\psi$, then $1+\lambda x \in G^\psi$ for all $\lambda \in \mathbb{F}_q$. This is clear if $x \in U$, so let $x \in A \setminus U$ be such that $1 + x \in G^\psi$, and let $N_x = 1 + U + \mathbb{F}_q \cdot x$ as in the proof of Theorem 2.1. Then N_x is an algebra group, and since $x \in G^\psi$, Mackey's criterion implies that $\operatorname{Ind}_H^{N_x} \psi$ is not irreducible.

Let ϕ be an irreducible summand of $\operatorname{Ind}_H^{N_x} \psi$; then $\dim(\psi) \leq \dim(\phi) < q \cdot \dim(\psi)$, whence $\dim \phi = \dim \psi$ by Corollary 2.2, and therefore $\psi \cong \phi\big|_H$ by Frobenius reciprocity. A fortiori, we have $N_x \subseteq G^\psi$, which completes the proof of the lemma. □

This also finishes the proof of Theorem 3.1. We should point out, however, that in practice it will be more convenient to use not the whole reduction process, but rather its first step. For instance, it has the advantage of producing algebra subgroups $G_i \subseteq G$ that contain $1 + A^2$, and, in particular, are normal.

3.3. We conclude this section by introducing two important invariants of irreducible representations of finite algebra groups. If $G = 1 + A$ is a finite algebra group over $\mathbb{F}_q$ and $\rho : G \to GL(V)$ is an irreducible representation, we define the *functional dimension* of ρ to be $\operatorname{fdim}(\rho) = \log_q(\dim_{\mathbb{C}} V)$. This definition is motivated by the geometric applications; note that $\operatorname{fdim}(\rho)$ is a nonnegative integer by Corollary 2.2. On the other hand, we define $\operatorname{sh}(\rho)$ to be the number of steps in the reduction process applied to ρ, with the understanding that $\operatorname{sh}(\rho) = 0$ if and only if ρ is strongly Heisenberg. We will see later on (in Theorem 6.1) that both of these invariants are stable under base change maps.

4. Base change maps for 1-dimensional representations

4.1. **Notation and terminology.** In this section we begin studying the aspects of character theory that have to do with extending the base field. All fields we consider throughout the rest of the paper are assumed to be finite subfields of $\mathbb{F}$, without any exceptions. If the "base field" is $\mathbb{F}_q$, and if $k = \mathbb{F}_{q^m}$ and $k' = \mathbb{F}_{q^n}$ are such that $k \subseteq k'$, i.e., $m|n$, then the base change map T_m^n will also be denoted by $T_k^{k'}$. (This is a slight abuse of notation, since in principle the base change maps may depend on the choice of the base field $\mathbb{F}_q$; however, as we will see, all our constructions depend only on the pair of fields $k \subseteq k'$.)

If $k = \mathbb{F}_q$, the topological generator Fr_q of the Galois group $\mathrm{Gal}(\mathbb{F}/k)$ will also be denoted by Fr_k. From now on, if A is a finite dimensional associative nilpotent algebra over k, we will say that $G = 1 + A$ is an *algebra group over* k, and we will no longer make an explicit distinction between finite algebra groups and unipotent algebra groups; we will also write $\dim G = \dim_k A$. If $k \subseteq k' \subseteq k''$ are field extensions, we will write $A' = k' \otimes_k A$, $A'' = k'' \otimes_k A = k'' \otimes_{k'} A'$, and we will write $G' = 1 + A'$, $G'' = 1 + A''$ for the algebra groups obtained by extending scalars from k to k' and k'', respectively.

Finally, if $\mathbb{F}_q \subseteq k$ and $G = 1 + A$ is an algebra group over k, we will say that A or G *is defined over* $\mathbb{F}_q$ if $A = k \otimes_{\mathbb{F}_q} A_0$ for some algebra A_0 over $\mathbb{F}_q$. We also make the convention that whenever a fixed power q of the prime p is present, we write Fr in place of Fr_q.

4.2. **Norm maps.** For any group G, we will denote by G^{ab} its abelianization, i.e., $G/(G, G)$. The main result of this section is the following

Theorem 4.1. *For all field extensions $k \subseteq k'$, and for all algebra groups $G = 1 + A$ over k, there exist group homomorphisms*

$$N_{k'/k}^A : G'^{ab} = \big(1 + A'\big)^{ab} \longrightarrow \big(1 + A\big)^{ab} = G^{ab}$$

with the following properties:

(1) $N_{k'/k}^A$ is functorial with respect to A (in the obvious sense);
(2) if A is defined over $\mathbb{F}_q \subseteq k$, then $N_{k'/k}^A$ is Fr*-equivariant;*
(3) if $k' \subseteq k''$ is another extension, we have

$$N_{k''/k}^A = N_{k'/k}^A \circ N_{k''/k'}^A;$$

(4) the homomorphism $N_{k'/k}^A$ is surjective.

The homomorphisms $N_{k'/k}^A$ of the theorem will be called the *norm maps*. Due to the functoriality property of the norm maps, there is little harm in omitting the superscript A from the notation, which we will do from now on.

Let us now make a few comments about the statement of the theorem, all of which follow easily from the proof given below. The construction of $N_{k'/k}$ depends only on k and k', and not on the field over which A is defined. Property (2) implies in particular that $N_{k'/k}$ is

invariant with respect to Fr_k, whence the kernel of $N_{k'/k}$ contains all elements of the form $\mathrm{Fr}_k(g)\cdot g^{-1}$, for $g\in G'$. Property (4) then implies that $N_{k'/k}$ yields a surjection

$$H_0\big(\mathrm{Gal}(k'/k),G'^{ab}\big)\twoheadrightarrow G^{ab}. \tag{4.1}$$

4.3. **Base change maps.** For any group G, we will denote by G^* the group of all homomorphisms $G\to\mathbb{C}^\times$, i.e., the group of isomorphism classes of one-dimensional representations of G. Of course, we have $G^*=(G^{ab})^*$. Moreover, since the functor $G\mapsto G^*$ is an anti-autoequivalence of the category of finite abelian groups, it is easy to see that the existence of norm maps with properties (1)–(3) of the theorem is equivalent to the existence of base change maps for one-dimensional representations of finite algebra groups, and property (4) is equivalent to the injectivity of the base change maps, since the latter are precisely the duals of the homomorphisms (4.1). These remarks will be used implicitly from now on.

As mentioned in the introduction, it is not known to us whether the base change maps $G^*\to(G'^*)^{\mathrm{Gal}(k'/k)}$ constructed here are surjective or not. However, since we are dealing with finite sets, it is clear that our base change maps are surjective if and only if $|G^{ab}|=|H_0(\mathrm{Gal}(k'/k),G'^{ab})|$ for all extensions $k\subseteq k'$ as above. Note that since all such extensions are cyclic, the last equality is equivalent to the equality

$$\big|G^{ab}\big|=\big|(G'^{ab})^{\mathrm{Gal}(k'/k)}\big|. \tag{4.2}$$

On the other hand, if our base change maps are not surjective, then, for the same reason, no surjective base change maps can possibly exist.

We will see in the next two sections that if the base change maps we have constructed are surjective for 1-dimensional representations, then they are surjective for arbitrary irreducible representations. Moreover, it is easy to see that our base change maps are necessarily surjective in some special cases. On the one hand, if the degree $[k':k]$ is relatively prime to p, then they are surjective because the obstruction to the injectivity of (4.1) lies in a certain cohomology group which vanishes in this case. On the other hand, if the group $G=1+A$ has nilpotence class less than p, then our base change maps have to be surjective, because there do exist surjective base change maps, constructed via the orbit method correspondence. If A satisfies the stronger condition $A^p=(0)$, then the exponential map $A\to 1+A$ is a well-defined bijection, and it is easy to check in this case that our base change maps coincide with the ones provided by the orbit method. However, if we only assume that G has nilpotence class less than p, then it is not clear to us how to describe the Lie ring scheme associated to G via Lazard's construction ([Khu97], [Dri05]) in terms of A, so we do not know if our base change maps coincide with the ones provided by the orbit method or not.

4.4. **Proof of Theorem 4.1.** The idea of the construction of the norm maps $N^A_{k'/k}$ was explained to the author by V. Drinfeld. Given an algebra group $G=1+A$ over k, let us write R for the k-algebra obtained by formally adjoining a unit to A. That is, R is the set of formal expressions of the form $\lambda\cdot 1+a$, where $\lambda\in k$ and $a\in A$, with the obvious algebra structure. We have an obvious canonical decomposition $R^\times=k^\times\times G$, which induces a decomposition $(R^\times)^{ab}=k^\times\times G^{ab}$. If we write $R'=k'\otimes_k R$, then we claim that it suffices to

define norm maps $(R'^{\times})^{ab} \to (R^{\times})^{ab}$ satisfying the properties stated in the theorem. Indeed, note that G'^{ab} is a p-group, whereas the order of $k^{\times}$ is relatively prime to p, so there can be no nontrivial homomorphisms from G'^{ab} to $k^{\times}$, and also from $k'^{\times}$ to G^{ab}. It follows immediately that norm maps for the groups $(R^{\times})^{ab}$ with properties (1)–(4) induce norm maps for algebra groups, satisfying the same properties.

In the remainder of the proof we will freely use the basic notions and results in algebraic K-theory (more precisely, we only need the functors K_0 and K_1). For all the missing explanations we refer the reader to [Sri96], Chapter 1, and [Weibel], Chapter III. It is proved in [Sri96], Example (1.6), that for any algebra R of the form considered above, the natural map $GL_1(R) \to K_1(R)$ induces an isomorphism $(R^{\times})^{ab} \xrightarrow{\simeq} K_1(R)$. On the other hand, if $R' = k' \otimes_k R$, as above, then R' is a projective (in fact, free) R-module of finite rank, so, as explained in [Weibel], Chapter III, there exists a natural "transfer map" $K_1(R') \to K_1(R)$. Explicitly, the transfer map is induced by the homomorphism $GL(R') \to GL(R)$ which comes from the fact that a matrix of size n over R' naturally gives rise to a matrix of size $n \cdot [k' : k]$ over R. It is clear from this description that the norm maps $(R'^{\times})^{ab} \to (R^{\times})^{ab}$ induced by the transfer maps in K-theory satisfy properties (1)–(3) of the theorem, and we only need to verify property (4).

To this end, we use an idea suggested by C. Weibel (private communication). If $R = k$, then the statement follows trivially from the multiplicative version of Hilbert's theorem 90. If $R \neq k$, we may assume by induction that the result holds for all algebras of smaller dimension. Let $I = \{a \in A \big| aA = Aa = (0)\}$. This is a nontrivial two-sided ideal of A, which leads to a natural exact sequence in K-theory (see, e.g., [Weibel], Proposition III.2.3)

$$K_1(R, I) \longrightarrow K_1(R) \longrightarrow K_1(R/I) \longrightarrow K_0(I) \tag{4.3}$$

Moreover, we clearly have $K_0(I) = (0)$, because I is nilpotent. Now the transfer maps yield a commutative diagram defining a morphism from a similar exact sequence

$$K_1(R', I') \longrightarrow K_1(R') \longrightarrow K_1(R'/I') \longrightarrow K_0(I') = (0) \tag{4.4}$$

to the sequence (4.3), where $I' = k' \otimes_k I$. The transfer map $K_1(R'/I') \to K_1(R/I)$ is surjective by the induction assumption. On the other hand, the description of $K_1(R, I)$ due to Vaserstein, explained in [Weibel], Exercise 2.4, implies that the natural map $1 + I \to K_1(R, I)$ is an isomorphism. Thus the transfer map $K_1(R', I') \to K_1(R, I)$ reduces to the natural norm map $1 + I' \to 1 + I$ for the commutative unipotent group $1 + I$, which is well known to be surjective (cf. [Lus03], or [Dri05], where the norm maps are called "trace maps" in the commutative case). Applying the precise version of the Five Lemma to the diagram formed by the sequences (4.3), (4.4) and the transfer maps, and taking into account the fact that the map $K_0(I') \to K_0(I)$ is injective for trivial reasons, we see that the transfer map $K_1(R') \to K_1(R)$ is surjective, completing the proof of Theorem 4.1.

5. Base change maps for strongly Heisenberg representations

5.1. In this section, using the norm maps for unipotent algebra groups constructed in Section 4, we will define base change maps for strongly Heisenberg representations, and prove their injectivity. If $G = 1 + A$ is a finite algebra group, we will write $\mathrm{SHeis}(G)$ for the set of isomorphism classes of strongly Heisenberg representations of G.

From now on we will use the following notation. If $\rho : 1+A_1 \to GL(V)$ is a representation of a finite algebra group over k and $f : A_1 \to A_2$ is an isomorphism of algebras over k, then $\rho\circ f^{-1}$ is a representation of $1+A_2$, which will be denoted by ρ^f. In particular, suppose that $1 + A_1$ is a normal algebra subgroup of a bigger algebra group $1 + A$. Then the conjugation by any element $g \in 1 + A$ induces an algebra automorphism of $1 + A_1$ whose action on representations of $1 + A_1$ will be denoted by $\rho \mapsto \rho^g$. The main result of this section is

Theorem 5.1. *For all field extensions $k \subseteq k'$, and for all algebra groups $G = 1 + A$ over k, there exist maps*

$$T_k^{k'} : \mathrm{SHeis}(G) \longrightarrow \mathrm{SHeis}(G')^{\mathrm{Gal}(k'/k)}$$

with the following properties:

(1) $T_k^{k'}$ is functorial with respect to isomorphisms of algebras over k, i.e., if $f : A_1 \to A_2$ is a k-algebra isomorphism and $f' : A_1' \to A_2'$ is obtained from f by extension of scalars, then $T_k^{k'}(\rho^f) = T_k^{k'}(\rho)^{f'}$ for all $\rho \in \mathrm{SHeis}(1 + A)$;

(2) if A is defined over $\mathbb{F}_q \subseteq k$, then $T_k^{k'}$ is Fr-equivariant, where $\mathrm{Fr} = \mathrm{Fr}_q$;

(3) if $k' \subseteq k''$ is another extension, we have

$$T_k^{k''} = T_{k'}^{k''} \circ T_k^{k'};$$

(4) the map $T_k^{k'}$ is injective;

(5) we have $\mathrm{fdim}(T_k^{k'}(\rho)) = \mathrm{fdim}(\rho)$ for all $\rho \in \mathrm{SHeis}(G)$;

(6) the map $T_k^{k'}$ is compatible with restriction of representations in the following sense. Suppose $K = 1+U$ is an algebra subgroup of G such that $1 + A^2 \subseteq K$, let $\rho \in \mathrm{SHeis}(G)$, and let $\psi \in \widehat{K}$ be an irreducible summand of $\rho\big|_K$. Then ψ is also strongly Heisenberg, and $T_k^{k'}(\psi) \in \mathrm{SHeis}(K')$ is an irreducible summand of $T_k^{k'}(\rho)\big|_{K'}$.

Moreover, we will also prove

Proposition 5.2. *If the base change maps for 1-dimensional representations defined in Section 4 are surjective for all algebra subgroups of G, then the base change maps for strongly Heisenberg representations of G are also surjective.*

5.2. To prove Theorem 5.1, we need to understand the structure of strongly Heisenberg representations. To this end, let $G = 1 + A$ be a finite algebra group over k, and let $\phi : 1 + A^2 \to \mathbb{C}^\times$ be a G-invariant homomorphism. Lemma 1.4 implies in particular that the kernel of the commutator pairing c_ϕ introduced in that lemma is a k-subspace of A/A^2, which implies that the group

$$G_\phi = \left\{ g \in G \,\middle|\, \phi(ghg^{-1}h^{-1}) = 1 \;\forall\, h \in G \right\}$$

is an algebra subgroup of G. It is not to be confused with the stabilizer G^{ϕ} of ϕ in G, which in our situation coincides with G. Moreover, c_{ϕ} induces a nondegenerate alternating bilinear form on the finite abelian group G/G_{ϕ}, and, as explained in the proof of Proposition 1.3, there exists a Lagrangian subgroup of G/G_{ϕ} which is also a k-subspace, i.e., has the form H/G_{ϕ} for an algebra subgroup $H \subseteq G$. By definition, this implies that ϕ is trivial on the commutator (H,H), whence ϕ admits a (non-unique) extension to a homomorphism $\psi : H \to \mathbb{C}^{\times}$. It is then an easy exercise to check that $\operatorname{Ind}_H^G \psi$ is irreducible, and in fact is a strongly Heisenberg representation of G. Moreover, it follows from Proposition 1.3 that every strongly Heisenberg representation of G is obtained in this way; more precisely, we have the following result, which follows immediately from the preceeding discussion and from the last statement of Lemma 1.1:

Lemma 5.3. *There is a natural bijection between the set of isomorphism classes of strongly Heisenberg representations of G whose restriction to $1+A^2$ is a multiple of ϕ and the set of extensions of ϕ to a homomorphism $\chi : G_{\phi} \to \mathbb{C}^{\times}$.*

Note that the definition of G_{ϕ} makes sense whenever we have a finite group G, a normal subgroup $K \subseteq G$ such that G/K is abelian, and a G-invariant character $\phi : K \to \mathbb{C}^{\times}$. The next result is also completely straightforward.

Lemma 5.4. *The following are equivalent for a subgroup $H \subseteq G$ such that $K \subseteq H$:*

(i) $H \subseteq G_{\phi}$;
(ii) ϕ can be extended to a G-invariant homomorphism $H \to \mathbb{C}^{\times}$;
(iii) ϕ extends to a homomorphism $H \to \mathbb{C}^{\times}$, and every such extension of ϕ is G-invariant.

5.3. We begin the proof of Theorem 5.1. Let G be a finite algebra group over k, let $k \subseteq k'$ be an extension, and let $\rho : G \to GL(V)$ be a strongly Heisenberg representation. By Lemmas 5.3 and 5.4, we have the corresponding pair (ϕ, χ), such that the restriction of ρ to $1+A^2$ is a multiple of ϕ, and χ is an extension of ϕ to G_{ϕ}, which is necessarily G-invariant. Using the norm maps defined in Section 4, we obtain characters $\widetilde{\phi} = \phi \circ N_{k'/k} : 1+A'^2 \to \mathbb{C}^{\times}$ and $\widetilde{\chi} = \chi \circ N_{k'/k} : (G_{\phi})' \to \mathbb{C}^{\times}$. We begin by observing that $\widetilde{\phi}$ is G'-invariant. Indeed, we know by Lemma 3.2 that the stabilizer of $\widetilde{\phi}$ in G' is an algebra subgroup of G'. On the other hand, if we let G act on A by conjugation, it follows from the functoriality of the norm maps that the induced action of G on A' leaves $\widetilde{\phi}$ invariant. Thus, if we view G as a subgroup of G' in the natural way, then $G \subseteq G'^{\widetilde{\phi}}$. As G' is the smallest algebra subgroup of G' containing G, this proves that $\widetilde{\phi}$ is G'-invariant.

By the same argument, we see that $\widetilde{\chi}$ is G'-invariant, and the functoriality of the norm maps implies that $\widetilde{\chi}$ is an extension of $\widetilde{\phi}$. Thus $(G_{\phi})' \subseteq (G')_{\widetilde{\phi}}$ by Lemma 5.4. To show that this inclusion is an equality, let us write $G_{\phi} = 1+U$, where U is a subspace, $A^2 \subseteq U \subseteq A$. Then $(G_{\phi})' = 1+U'$. On the other hand, $(G')_{\widetilde{\phi}}$ is invariant under the action of $\operatorname{Gal}(k'/k)$ on G', and is also an algebra subgroup of G', whence $(G')_{\widetilde{\phi}} = 1+V'$ for some subspace V such that $U \subseteq V \subseteq A$. Suppose, for the sake of contradiction, that $U \neq V$. Then there exists a

subspace $W \subseteq V$ such that $U \subsetneq W$ and such that ϕ admits an extension $\lambda : 1 + W \to \mathbb{C}^\times$. Hence $\widetilde{\lambda} = \lambda \circ N_{k'/k} : 1 + W' \to \mathbb{C}^\times$ is an extension of $\widetilde{\phi}$. Since $W' \subseteq V'$, it follows from Lemma 5.4 that $\widetilde{\lambda}$ is G'-invariant, and hence, a fortiori, G-invariant. Now the surjectivity and the G-equivariance of the norm map $N_{k'/k} : (1 + W')^{ab} \to (1 + W)^{ab}$ implies that λ is also G-invariant. However, by Lemma 5.4, this contradicts the choice of W.

The conclusion of the last two paragraphs is that the pair $(\widetilde{\phi}, \widetilde{\chi})$ satisfies the assumptions of Lemma 5.3, and hence gives rise to a strongly Heisenberg representation $\widetilde{\rho}$ of G', determined uniquely up to isomorphism. It is clear that since $\widetilde{\phi}$ and $\widetilde{\chi}$ are $\mathrm{Gal}(k'/k)$-invariant, so is ρ, whence we have defined the base change maps $T_k^{k'}$ for strongly Heisenberg representations as stated in Theorem 5.1. Property (5) is obvious from the construction, since if $\rho \in \mathrm{SHeis}(G)$ corresponds to a pair (ϕ, χ) as in Lemma 5.3, then $\mathrm{fdim}(\rho) = \dim_k(G/G_\phi)/2$, and we have $\dim_k(G/G_\phi) = \dim_{k'}(G'/(G')_{\widetilde{\phi}})$ by the argument given above. Properties (1)–(3) also follow trivially from the construction and the corresponding properties of the norm maps.

5.4. Let us prove property (4). Suppose we have two strongly Heisenberg representations, ρ_1 and ρ_2, of G, corresponding to pairs (ϕ_1, χ_1) and (ϕ_2, χ_2), such that $\widetilde{\rho_1} \cong \widetilde{\rho_2}$. Using the construction of the base change maps and Lemma 5.3, we see immediately that $\widetilde{\phi_1} = \widetilde{\phi_2}$, which forces $\phi_1 = \phi_2$ by the surjectivity of the norm maps. Then, a fortiori, we have $(G')_{\widetilde{\phi_1}} = (G')_{\widetilde{\phi_2}}$, whence $\widetilde{\rho_1} \cong \widetilde{\rho_2}$ forces $\widetilde{\chi_1} = \widetilde{\chi_2}$ by Lemma 5.3, which, in turn, forces $\chi_1 = \chi_2$ by the surjectivity of the norm maps.

Finally, we prove property (6). Once again, let (ϕ, χ) be the pair of linear characters corresponding to ρ. Since ψ is a summand of $\rho\big|_K$, it follows that ϕ is a summand of $\psi\big|_{1+A^2}$. But ϕ is invariant under G, and hence, a fortiori, under K; since ψ is irreducible, we see that $\psi(1 + A^2)$ consists only of scalar operators, whence ψ is also strongly Heisenberg, and acts by the character ϕ on $1 + A^2$. Furthermore, it is clear that $1 + A^2 \subseteq G_\phi \subseteq K_\phi$. Let λ denote the extension of ϕ to K_ϕ which corresponds to the representation ψ as in Lemma 5.3. Then $\psi\big|_{G_\phi}$ is a multiple of $\lambda\big|_{G_\phi}$ and $\rho\big|_{G_\phi}$ is a multiple of χ, which implies that λ is an extension of χ. Now the functoriality of the norm maps implies that $\widetilde{\lambda}$ is an extension of $\widetilde{\chi}$, and, reversing the argument above, we see that $\widetilde{\psi}$ is a summand of $\widetilde{\rho}\big|_{K'}$, which completes the proof of Theorem 5.1.

5.5. We conclude by proving Proposition 5.2. Let ρ' be a $\mathrm{Gal}(k'/k)$-invariant strongly Heisenberg representation of G', and let (ϕ', χ') be the corresponding pair of linear characters. Then both ϕ' and χ' are $\mathrm{Gal}(k'/k)$-invariant. The assumption of the proposition implies that $\phi' = \widetilde{\phi}$ for some character $\phi : 1 + A^2 \to \mathbb{C}^\times$. Moreover, the G-invariance and surjectivity of the norm maps implies that ϕ is G-invariant. By the previous argument, $(G')_{\phi'} = (G_\phi)'$. Since χ' is $\mathrm{Gal}(k'/k)$-invariant, we have $\chi' = \widetilde{\chi}$ for some character $\chi : G_\phi \to \mathbb{C}^\times$, by assumption. Moreover, surjectivity of the norm maps implies that χ is an extension of ϕ. Now let ρ be the (unique) strongly Heisenberg representation of G defined by the pair (ϕ, χ); we have $\rho' \cong \widetilde{\rho}$ by construction, which completes the proof.

6. Base change maps for general irreducible representations

6.1. We are now ready to define base change maps for arbitrary irreducible representations of algebra groups. Let $k \subseteq k'$ be an extension of finite subfields of $\mathbb{F}$, as usual; let $G = 1 + A$ be a finite algebra group over k, and let $\rho : G \to GL(V)$ be an irreducible representation. We define a representation $\widetilde{\rho}$ of G', which a priori may not be irreducible, inductively, as follows. If ρ is strongly Heisenberg, we set $\widetilde{\rho} = T_k^{k'}(\rho)$, where $T_k^{k'}$ is defined in the previous section. If ρ is not strongly Heisenberg, let

$$V = \bigoplus_{i \in I} V_i \tag{6.1}$$

be the canonical decomposition of V obtained in the first step of the reduction process of Section 3. Pick $i \in I$, and let ρ_i denote the representation of G_i in V_i. Then $\mathrm{sh}(\rho_i) = \mathrm{sh}(\rho) - 1$ (cf. the end of Section 3), so by induction, we may assume that $\widetilde{\rho_i}$ has been defined (though it may not be irreducible). We set

$$\widetilde{\rho} = \mathrm{Ind}_{G_i'}^{G'} \widetilde{\rho_i}. \tag{6.2}$$

The main result of this section is the following

Theorem 6.1. *(1) The representation $\widetilde{\rho}$ defined by* (6.2) *is irreducible.*

(2) Up to isomorphism, $\widetilde{\rho}$ is independent of the choice of i. Thus it is legitimate to define

$$T_k^{k'}(\rho) = \widetilde{\rho},$$

(3) The map $\rho \mapsto \widetilde{\rho}$ is functorial with respect to isomorphisms of algebras over k; moreover, if A is defined over $\mathbb{F}_q \subseteq k$, then this map commutes with Fr_q. A fortiori, it commutes with Fr_k, whence, in view of (1) and (2), we obtain base change maps

$$T_k^{k'} : \widehat{G} \longrightarrow \left(\widehat{G'}\right)^{\mathrm{Gal}(k'/k)}, \qquad \rho \mapsto \widetilde{\rho}, \tag{6.3}$$

which are independent of any choices.

(4) Suppose $K \subseteq G$ is an algebra subgroup such that $1 + A^2 \subseteq K$, let $\rho \in \widehat{G}$, and let $\psi \in \widehat{K}$ be an irreducible summand of $\rho\big|_K$. Then $\widetilde{\psi}$ is an irreducible summand of $\widetilde{\rho}$.

(5) If $k' \subseteq k''$ is another extension, then

$$T_k^{k''} = T_{k'}^{k''} \circ T_k^{k'}.$$

(6) The base change map (6.3) *is injective.*

(7) We have $\mathrm{fdim}(\widetilde{\rho}) = \mathrm{fdim}(\rho)$ *for all $\rho \in \widehat{G}$.*

(8) We have $\mathrm{sh}(\widetilde{\rho}) = \mathrm{sh}(\rho)$ *for all $\rho \in \widehat{G}$.*

It is clear that this result implies the theorem stated in the introduction. The reason for stating Theorem 6.1 in this form is that, as will become apparent, the interplay between the various parts of the theorem is essential for the inductive proof we present below.

6.2. The theorem will be proved simultaneously for all field extensions $k \subseteq k' \subseteq k''$ by induction on $\dim G$ and on $\mathrm{sh}(\rho)$. When $\mathrm{sh}(\rho) = 0$, it is clear that all the non-vacuous statements of Theorem 6.1 follow from Theorem 5.1, so we may assume that $\mathrm{sh}(\rho) > 0$ and that Theorem 6.1 holds for irreducible representations ϕ of arbitrary finite algebra groups and for arbitrary extensions $k \subseteq k' \subseteq k''$, as long as $\mathrm{sh}(\phi) < \mathrm{sh}(\rho)$. Similarly, we assume that the theorem holds for all algebra groups of dimension less than $\dim_k A$.

We begin by proving parts (1), (2) and (3). Since $\mathrm{sh}(\rho_i) < \mathrm{sh}(\rho)$, the theorem holds for the group G_i by assumption, and, in particular, $\widetilde{\rho}_i$ is irreducible. By Mackey's criterion, to show that $\widetilde{\rho}$ is irreducible, we must check that the stabilizer $G'^{\widetilde{\rho}_i}$ of (the isomorphism class of) $\widetilde{\rho}_i$ in G' equals G'_i. By Lemma 3.2, $G'^{\widetilde{\rho}_i}$ is an algebra subgroup of G'; moreover, it is clearly $\mathrm{Gal}(k'/k)$-invariant, because $\widetilde{\rho}_i$ is $\mathrm{Gal}(k'/k)$-invariant by part (3) of the theorem. Thus, if $G'^{\widetilde{\rho}_i} \neq G'_i$, then there exists an element $g \in G \setminus G_i$ such that $g \in G'^{\widetilde{\rho}_i}$ (as usual, we view G as a subgroup of G'). But conjugation by g induces an algebra automorphism of the k-subalgebra of A corresponding to G_i. Since the base change map for G_i is injective and commutes with the action of g by the induction assumption, it follows that $g \in G^{\rho_i}$, which contradicts the irreducibility of $\rho \cong \mathrm{Ind}_{G_i}^{G} \rho_i$, by Mackey's criterion. This proves part (1) for the group G.

Part (2) is easy, since, by construction, for any two elements $i, j \in I$, there exists an element $g \in G$ which conjugates the pair (G_i, ρ_i) into (G_j, ρ_j). Applying part (3) to G_i, we see that g also conjugates $(G'_i, \widetilde{\rho}_i)$ into $(G'_j, \widetilde{\rho}_j)$, whence

$$\mathrm{Ind}_{G'_i}^{G'} \widetilde{\rho}_i \cong \mathrm{Ind}_{G'_j}^{G'} \widetilde{\rho}_j,$$

proving (2). For part (3), we recall that the first step of the reduction process of Section 3 is functorial with respect to algebra isomorphisms by construction, which implies that if the base change maps for the groups G_i are functorial with respect to algebra isomorphisms, then so are the base change maps for G. A similar argument implies that if A is defined over $\mathbb{F}_q$, then $T_k^{k'}$ commutes with $\mathrm{Fr} = \mathrm{Fr}_q$, which proves (3).

6.3. To prove part (4), we first claim that it suffices to consider the case where K has codimension 1 in G. Indeed, if $K = G$, there is nothing to prove. Otherwise, there exists an algebra subgroup $K_0 \subset G$ of codimension 1 such that $K \subseteq K_0$. Since ψ is a summand of $\rho\big|_K$, it follows from Frobenius reciprocity that there exists an irreducible representation ϕ of K_0 that is a summand of both $\mathrm{Ind}_{K_0}^{K} \psi$ and ρ_{K_0}. If we know part (4) for the codimension 1 case, then $\widetilde{\phi}$ is a summand of $\widetilde{\rho}\big|_{K'_0}$. On the other hand, ψ is a summand of $\phi\big|_K$ by Frobenius reciprocity, and since $\dim K_0 < \dim G$, it follows from the induction assumption that $\widetilde{\psi}$ is a summand of $\widetilde{\phi}\big|_{K'}$. Combining these two statements yields the desired conclusion.

Now we consider the case where $\dim(G/K) = 1$. Recall that $\mathrm{sh}(\rho) > 0$ by assumption, and consider the decomposition (6.1) obtained in the first step of the reduction process for ρ. We fix $i \in I$ and let $\rho_i : G_i \to GL(V_i)$ be the corresponding representation of G_i. There

are two possibilities. If $K \supseteq G_i$, then ψ is a summand of

$$\rho\big|_K = \operatorname{Res}^G_K \operatorname{Ind}^G_K \big(\operatorname{Ind}^K_{G_i} \rho_i\big).$$

Let us write $\phi_i = \operatorname{Ind}^K_{G_i} \rho_i$, which is also irreducible. By Mackey's criterion, $\rho\big|_K$ is a direct sum of pairwise nonisomorphic irreducible representations, all of which are conjugate to ϕ_i. Hence ψ is conjugate to ϕ_i. But since we are allowed to replace (G_i, ρ_i) with any pair conjugate to it (which is of the form (G_j, ρ_j) for some $j \in I$), and since G_i is normal in G (because it contains $1 + A^2$), we may assume, without loss of generality, that $\psi \cong \phi_i$. By the induction assumption, $\widetilde{\rho}_i$ is a summand of $\widetilde{\psi}\big|_{G'_i}$. By the Frobenius reciprocity,

$$\big[\widetilde{\psi} : \widetilde{\rho}\big|_{K'}\big] = \big[\operatorname{Res}^{G'}_{G'_i} \operatorname{Ind}^{G'}_{K'} \widetilde{\psi} : \widetilde{\rho}_i\big],$$

and since $\operatorname{Res}^{G'}_{K'} \operatorname{Ind}^{G'}_{K'} \widetilde{\psi}$ certainly contains $\widetilde{\psi}$ as a summand, we obtain that $[\widetilde{\psi} : \widetilde{\rho}\big|_{K'}] > 0$, as desired.

The only remaining case is $K \not\supseteq G_i$, which forces $G = KG_i$. In this case it is elementary to check that

$$\operatorname{Res}^G_K \operatorname{Ind}^G_{G_i} \rho_i \cong \operatorname{Ind}^K_{K\cap G_i} \operatorname{Res}^{G_i}_{K\cap G_i} \rho_i,$$

using only the definition of an induced representations. Thus, Frobenius reciprocity implies that if ψ is a summand of $\rho\big|_K$, then $[\psi\big|_{K\cap G_i} : \rho_i\big|_{K\cap G_i}] > 0$. Let ϕ be any irreducible representation of $K \cap G_i$ contained both in $\psi\big|_{K\cap G_i}$ and in $\rho_i\big|_{K\cap G_i}$. By the induction assumption, it follows that $\widetilde{\phi}$ is contained both in $\widetilde{\phi}\big|_{K'\cap G'_i}$ and in $\widetilde{\rho}_i\big|_{K'\cap G'_i}$. Reversing the argument above, we find that $\widetilde{\psi}$ is contained in $\widetilde{\rho}\big|_{K'}$, completing the proof of (4).

6.4. Before proceeding with the proof, we need the following

Lemma 6.2. *In the situation described before the statement of Theorem 6.1, the pair $(G'_i, \widetilde{\rho}_i)$ is among those that appear in the first step of the reduction process applied to the representation $\widetilde{\rho}$ of G'.*

Proof. By construction, the pair (G_i, ρ_i) is obtained in the following way. Let ψ_i be an irreducible summand of $\rho\big|_{1+A^2}$ corresponding to $i \in I$, let $V_i \subseteq V$ be the sum of all subrepresentations of $\rho\big|_{1+A^2}$ that are isomorphic to ψ_i, let $G_i = G^{\psi_i}$, and let ρ_i denote the representation of G_i in V_i. Now by part (4), which we have already proved, $\widetilde{\psi}_i$ is an irreducible summand of $\widetilde{\rho}\big|_{1+A'^2}$, so we might as well use $\widetilde{\psi}_i$ in the first step of the reduction process applied to $\widetilde{\rho}$. On the other hand, we have $G_i \subseteq (G')^{\widetilde{\psi}_i}$ by part (3) of the theorem, whence $G'_i \subseteq (G')^{\widetilde{\psi}_i}$ by Lemma 3.2.

Conversely, since $(G')^{\widetilde{\psi}_i}$ is an algebra subgroup of G' and is clearly $\operatorname{Gal}(k'/k)$-stable, we have $G'^{\widetilde{\psi}_i} = H'$ for some algebra subgroup $H \subseteq G$. Since the base change maps for $1 + A^2$ are injective by the induction hypothesis, we deduce that $H \subseteq G^{\psi_i} = G_i$, which implies that $(G')^{\widetilde{\psi}_i} = G'_i$. On the other hand, we have $\operatorname{Ind}^{G'}_{G'_i} \widetilde{\rho}_i = \widetilde{\rho}$ by construction, and $\widetilde{\rho}_i\big|_{1+A'^2}$ contains

$\widetilde{\psi}_i$ as a summand by part (4). Since $\widetilde{\psi}_i$ is G'_i-invariant, also by construction, it follows that $\widetilde{\rho}_i\big|_{1+A'^2}$ is a sum of copies of $\widetilde{\psi}_i$. Finally, since $\operatorname{Ind}_{G'_i}^{G'}\widetilde{\rho}_i = \widetilde{\rho}$ is irreducible, this forces $\widetilde{\rho}_i$ to be the representation of G_i in the sum of *all* irreducible subrepresentations of $\widetilde{\rho}\big|_{1+A'^2}$ that are isomorphic to $\widetilde{\psi}_i$ and completes the proof of the lemma. □

6.5. We proceed with the proof of the theorem. Part (5) follows trivially from the previous lemma and the induction hypothesis. Moreover, Lemma 6.2 clearly implies (8) by induction on $\operatorname{sh}(\rho)$, which then implies (7) by induction. It remains to prove part (6). Suppose $\rho, \phi \in \widehat{G}$ are such that $\widetilde{\rho} \cong \widetilde{\phi}$. Then $\operatorname{sh}(\rho) = \operatorname{sh}(\phi)$ by part (8), and we will prove that $\rho \cong \phi$ by induction on $\operatorname{sh}(\rho)$, the case $\operatorname{sh}(\rho) = 0$ being established in Theorem 5.1.

Suppose that $\operatorname{sh}(\rho) = \operatorname{sh}(\phi) > 0$. Since $\widetilde{\rho} \cong \widetilde{\phi}$, and since the first step of the reduction process of Section 3 is canonically defined, it follows from Lemma 6.2 that at least the subgroups $G_i \cong G$ appearing in the first step of the reduction process are the same for ρ and ϕ. Let ρ_i, ϕ_i denote the corresponding irreducible representations of G_i, so that, in particular, $\rho \cong \operatorname{Ind}_{G_i}^{G}\rho_i$ and $\phi \cong \operatorname{Ind}_{G_i}^{G}\phi_i$. By definition, we now have

$$\operatorname{Ind}_{G'_i}^{G'}\widetilde{\rho}_i \cong \widetilde{\rho} \cong \widetilde{\phi} \cong \operatorname{Ind}_{G'_i}^{G'}\widetilde{\phi}_i,$$

which implies, by Frobenius reciprocity, that $\widetilde{\phi}_i$ is contained in $\widetilde{\rho}\big|_{G'_i}$. However, by Mackey's criterion, $\widetilde{\rho}\big|_{G'_i}$ is a direct sum of pairwise nonisomorphic representations of G'_i that are G'-conjugate to $\widetilde{\rho}_i$. We deduce that there exists a $g \in G'$ which conjugates $\widetilde{\rho}_i$ into $\widetilde{\phi}_i$.

Recall that Fr_k denotes the canonical topological generator of $\operatorname{Gal}(\mathbb{F}/k)$. Since $\widetilde{\rho}_i$ and $\widetilde{\phi}_i$ are $\operatorname{Gal}(k'/k)$-invariant, it follows that $\operatorname{Fr}_k(g)$ also conjugates $\widetilde{\rho}_i$ into $\widetilde{\phi}_i$. Hence $\operatorname{Fr}_k(g)^{-1}\cdot g \in (G')^{\widetilde{\rho}_i} = G'_i$. By Lang's theorem, there exists a finite extension $k'' \supseteq k'$ and an element $y \in G''_i$ such that $\operatorname{Fr}_k(y)\cdot y^{-1} = \operatorname{Fr}(g^{-1})\cdot g$. We then have $gy \in G$; on the other hand, by construction, gy conjugates $T_k^{k''}(\rho_i)$ into $T_k^{k''}(\phi_i)$. Applying parts (3) and (6) to the group G_i, we see that gy conjugates ρ_i into ϕ_i, which forces $\rho \cong \phi$ and completes the proof.

6.6. We now prove the following

Proposition 6.3. *If $G = 1 + A$ is an algebra group over k such that the base change maps for 1-dimensional representations defined in Section 4 are surjective for all algebra subgroups of G and all extensions $k' \subseteq k''$ containing k, then the base change for arbitrary irreducible representations of G are also surjective.*

Proof. Let ϕ be any $\operatorname{Gal}(k'/k)$-invariant irreducible representation of G'. We will prove that $\phi \cong \widetilde{\rho}$ for some $\rho \in \widehat{G}$ by induction on $\operatorname{sh}(\phi)$, the case $\operatorname{sh}(\phi) = 0$ being established in Proposition 5.2. Assume that $\operatorname{sh}(\phi) > 0$, and let ψ be any irreducible summand of $\phi\big|_{1+A'^2}$. Then so is $\operatorname{Fr}_k(\psi)$, whence there exists an element $g \in G'$ such that $\psi^g \cong \operatorname{Fr}_k(\psi)$. Here, for an irreducible representation λ of $1 + A'$ and an element $y \in G'$, we denote by λ^y the representation of $1 + A'$ obtained by conjugating λ by y. By Lang's theorem, there exists a

finite extension $k'' \supseteq k'$ and an element $x \in G''$ such that $g = \mathrm{Fr}_k(x)\cdot x^{-1}$. A straightforward computation yields

$$\mathrm{Fr}_k\big(T_{k'}^{k''}(\psi)^{x^{-1}}\big) \cong T_{k'}^{k''}(\psi)^{x^{-1}}.$$

Thus $T_{k'}^{k''}(\psi)^{x^{-1}}$ is a Fr_k-invariant summand of $T_{k'}^{k''}(\phi)\big|_{1+A''^2}$.

Now we may assume, without loss of generality, that $k' = k''$ and that ψ is already Fr_k-invariant. Indeed, if we can prove that $T_{k'}^{k''}(\phi) \cong T_k^{k''}(\rho)$ for some $\rho \in \widehat{G}$, then the injectivity of the base change maps, proved in Theorem 6.1, will imply that $\phi \cong T_k^{k'}(\rho)$. With this assumption, let us apply the first step of the reduction process to ϕ and this particular choice of ψ. It clearly yields a pair of the form (G_i', ϕ_i), where $G_i' = G'^{\psi}$ is obtain by extension of scalars from an algebra subgroup $G_i \subseteq G$. Moreover, since G_i is normal in G (and hence is the same for all i), and all possible choices for ϕ_i are G'-conjugate to each other, the same argument as above using Lang's theorem shows that we may assume that ϕ_i is also $\mathrm{Gal}(k'/k)$-invariant, possibly after replacing k' with a finite extension $k'' \supseteq k'$.

In this case, since $\mathrm{sh}(\phi_i) = \mathrm{sh}(\phi) - 1$, we can write $\phi_i = \widetilde{\rho_i}$ for some $\rho_i \in \widehat{G}_i$ by the induction assumption. The standard argument using Mackey's criterion implies then that $\rho := \mathrm{Ind}_{G_i}^{G} \rho_i$ is irreducible. In addition, part (4) of the theorem implies that $\phi_i = \widetilde{\rho_i}$ is a summand of $\widetilde{\rho}\big|_{G_i'}$, whence $\phi \cong \widetilde{\rho}$ by Frobenius reciprocity, completing the proof. $\square$

This result has the following curious consequence, whose conclusion is independent of the construction of base change maps, but which we were unable to verify directly. Of course, if one can find an example where the conclusion of this corollary does not hold, it would imply that the base change maps for 1-dimensional representations of algebra groups are not necessarily surjective. The proof is completely straightforward and therefore omitted.

Corollary 6.4. *Let $G = 1 + A$ be an algebra group over k satisfying the assumption of Proposition 6.3, let $H \subseteq G$ be an algebra subgroup such that $1 + A^2 \subseteq H$, and let $k \subseteq k'$ be an extension. If $\phi \in \widehat{G}'$ is $\mathrm{Gal}(k'/k)$-invariant, then $\phi\big|_{H'}$ has a $\mathrm{Gal}(k'/k)$-invariant irreducible summand.*

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